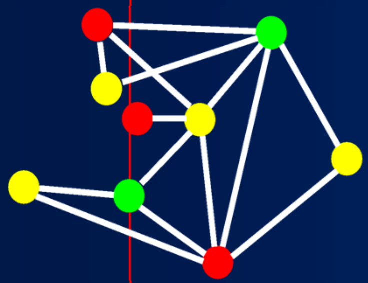
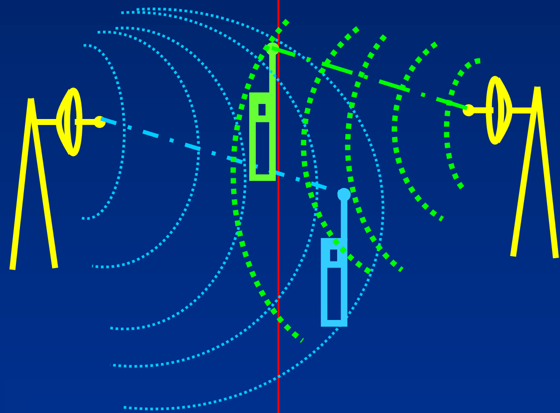
Graph Colouring and Frequency Assignment



Martin Grötschel

Andreas Eisenblätter

Arie M. C. A. Koster



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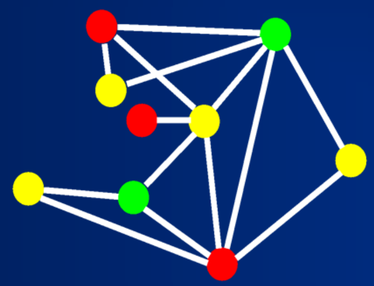
Coloring Graphs

Given a graph G = (V,E), color the nodes of the graph such that no two adjacent nodes have the same color.



The smallest number of colors with this property is called chromatic or coloring number and is denoted by $\chi(G)$.

Coloring Graphs

A typical theoretical question: Given a
class *C* of graphs
(e.g., planar or perfect graphs, graphs without certain minors), what can one prove about the chromatic number of all graphs in *C*?

A typical practical question: Given a
particular graph G
(e.g., arising in some application), how can one determine (or approximate) the chromatic number of G?

Coloring Graphs: Some History

- 1852 Francis Guthrie: Can any map be colored using four colors?

  1879 “False Proof” by Kempe

  1880 “False Proof” by Tait

  etc.

- 1890 Heawood “Map Color Theorem”

  (e. g., 7 colors suffice on the torus)

  1966 Proof by Ringel & Youngs, see

  G. Ringel: Map Color Theorem”, Springer, 1974

Coloring Graphs

- The Four Color Problem

  Appel & Haken (1977) "Proof" (Heesch)

  Appel & Haken (1986):

This leaves the reader to face 50 pages containing text and diagrams, 85 pages filled with almost 2500 additional diagrams, and 400 microfiche pages that contain further diagrams and thousands of individual verifications of claims made in 24 lemmas in the main section of the text. In addition the reader is told that certain facts have been verified with the use of about 1200 hours of computer time and would be extremely time-consuming to verify by hand. The papers are somewhat intimidating due to their style and length and few mathematicians have read them in any detail.

Coloring Graphs

- The Four Color Problem

Robertson, Sanders, Seymour & Thomas (1997)
(on the run: coloring algorithm with quadratic running time)

Robin Thomas: “An update of the four-color theorem”, Notices Amer. Math. Soc. 45 (1998) 848-859

http://www.math.gatech.edu/~thomas/FC/fourcolor.html

Coloring Graphs

- Random graph theory

  Very precise estimates for the chromatic number of random graphs

- Variations

  Edge coloring: Vizing's Theorem (1964)

  List coloring

  T-coloring

Coloring Graphs

- Coloring, T-coloring, and list coloring graphs embedded on surfaces
  Carsten Thomassen

Coloring Graphs

- Coloring graphs algorithmically

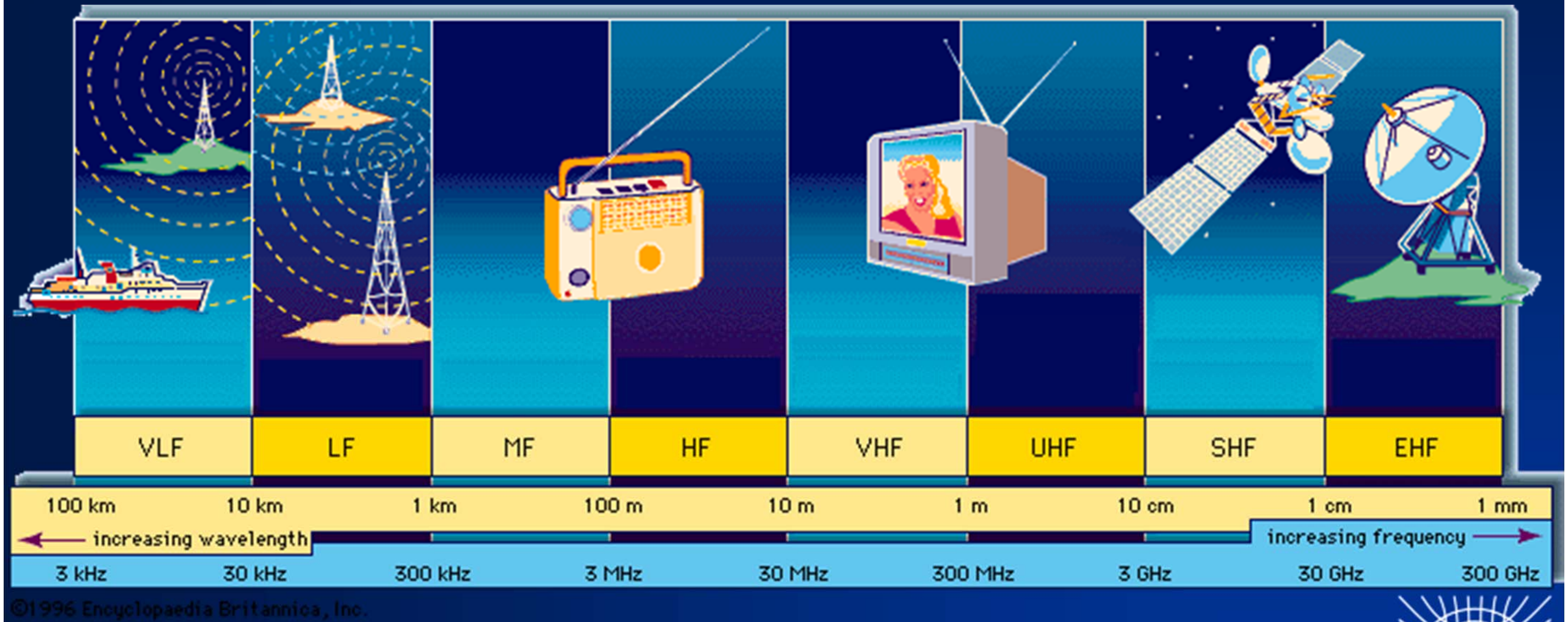
  - NP-hard in theory
  - very hard in practice
  - almost impossible to find optimal colorings
  - playground for heuristics (e.g., DIMACS challenge)

Coloring in Telecommunication

- Frequency or Channel Assignment for radio-, tv-transmission, etc.
- Our Example: Mobile phone systems

- Andreas Eseinblätter, Martin Grötschel and Arie M. C. A. Koster, *Frequenzplanung im Mobilfunk*, DMV-Mitteilungen 1(2002)18-25
- Andreas Eisenblätter, Hans-Florian Geerdes, Thorsten Koch, Ulrich Türke: *MOMENTUM Data Scenarios for Radio Network Planning and Simulation* , ZIB-Report 04-07
- Andreas Eisenblätter, Armin Fügenschuh, Hans-Florian Geerdes, Daniel Junglas, Thorsten Koch, Alexander Martin: *Optimization Methods for UMTS Radio Network Planning*, ZIB-Report 03-41

Wireless Communication



Wireless Communication

Current Mobile Phone Standards:

GSM (Group Spéciale Mobile, General System for Mobile Communication)

- since 1992 (Europe, Australia, parts of Asia, also spreadinging in the US)

CDMA (Code Division Multilpe Access)

- since 1995 (North America, parts of Asia)

GPRS (General Packet Radio System)

- since 2001 (in Germany)

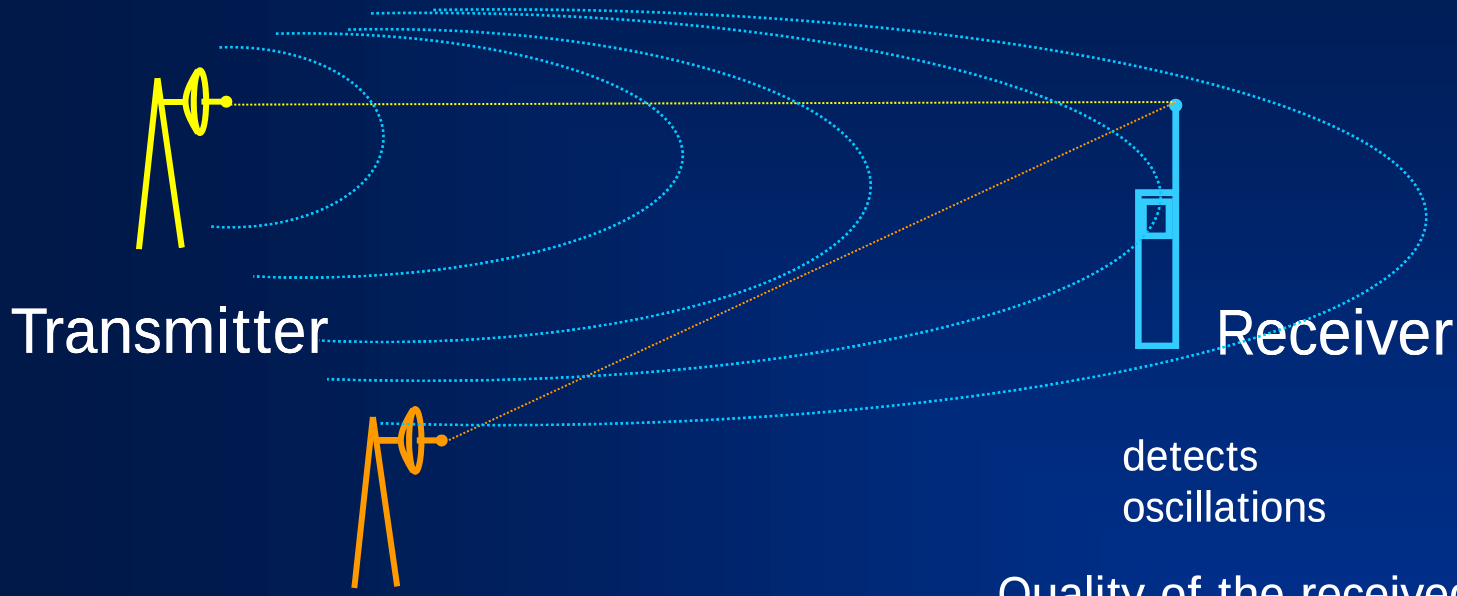
UMTS not covered in this talk

Wireless Communication

GSM Frequencies: 450, 900, 1800, 1900 MHz

More than 500 million users in over 150 countries

Properties of wireless communication



detects oscillations

emits electromagnetic oscillations at a frequency

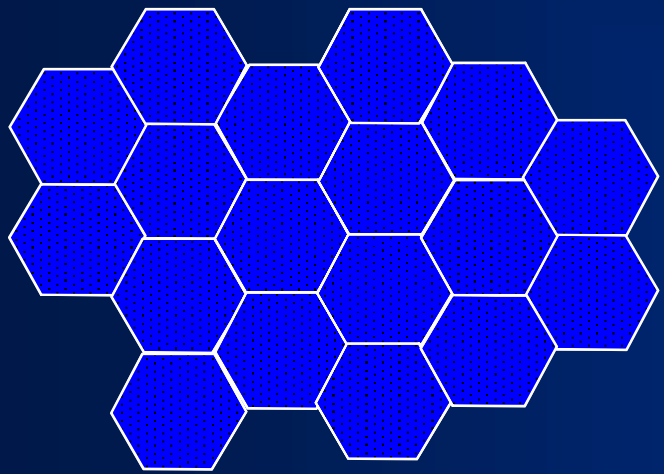
Quality of the received signal:
Signal-to-noise ratio

Poor signal-to-noise ratio:
interference of the signal

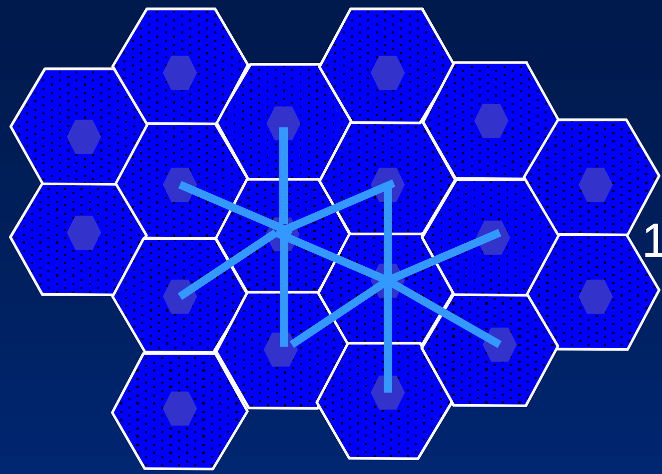
Objective: Frequency plan without interference or, second best, with minimum interference

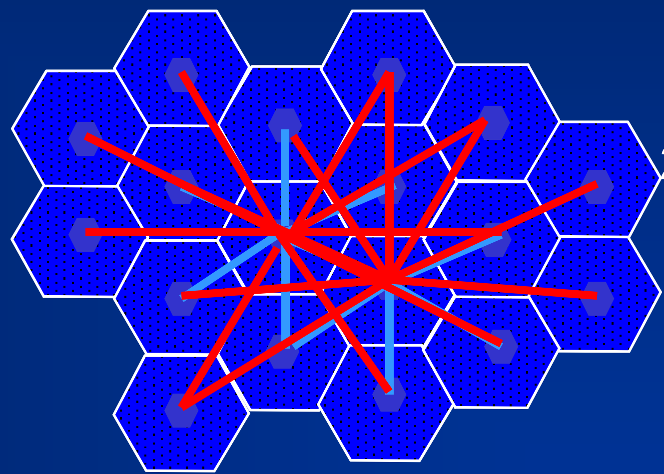
Cell Models

Hexagon Cell Model

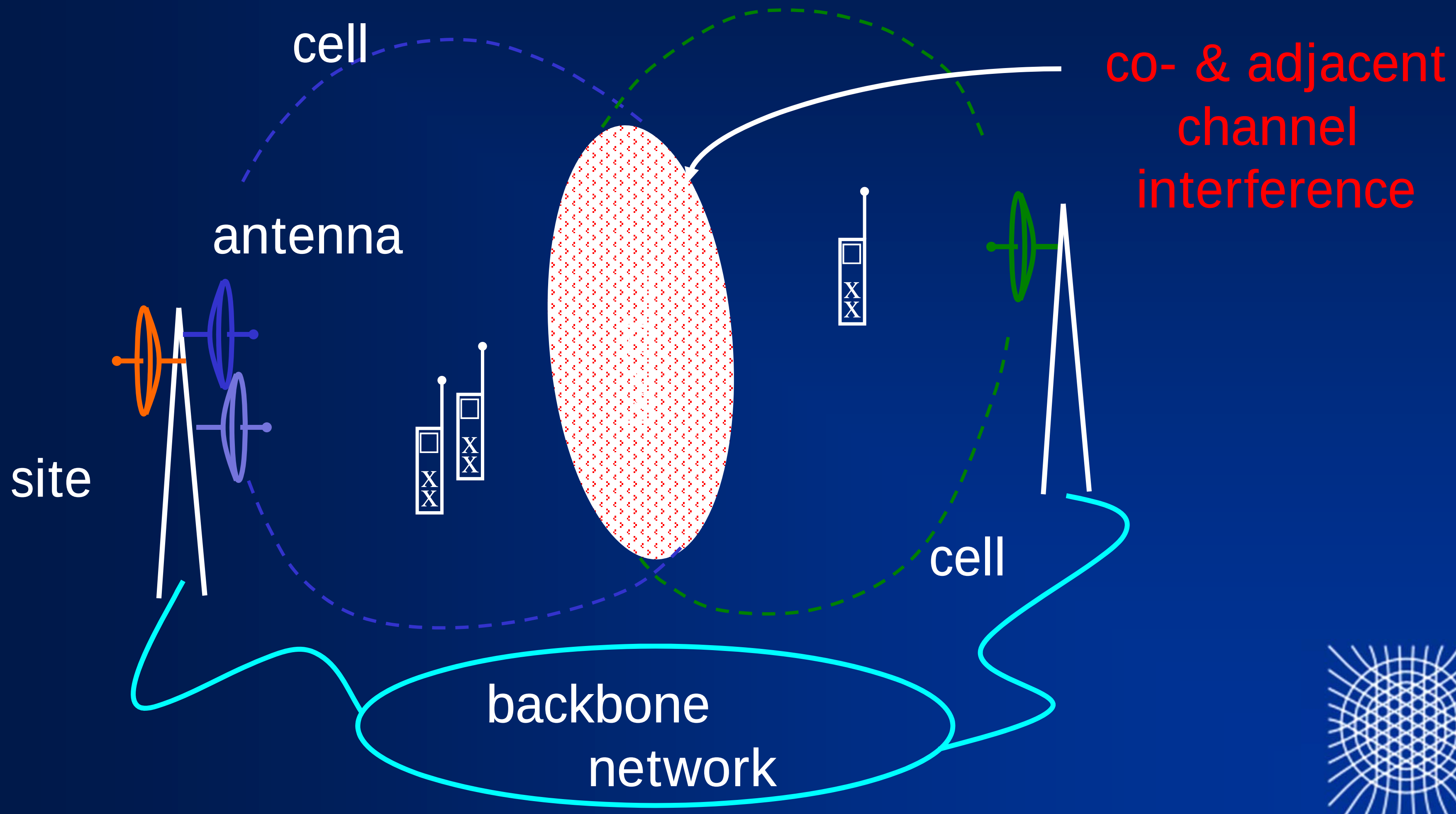


- **sites on regular grid**
- **isotropic propagation conditions**
- **no cell-overlapping**



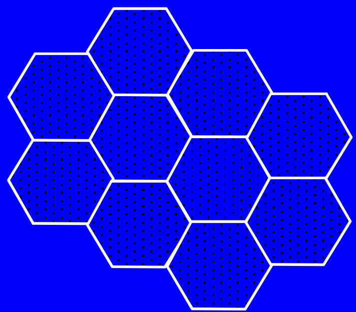


Antennas & Interference



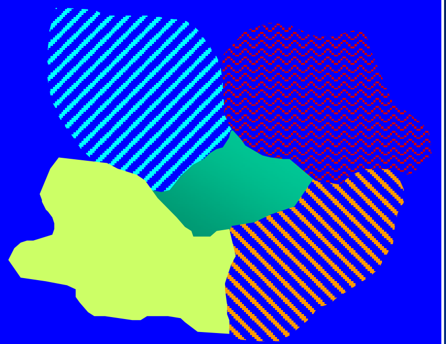
Cell Models

Hexagon Cell Model



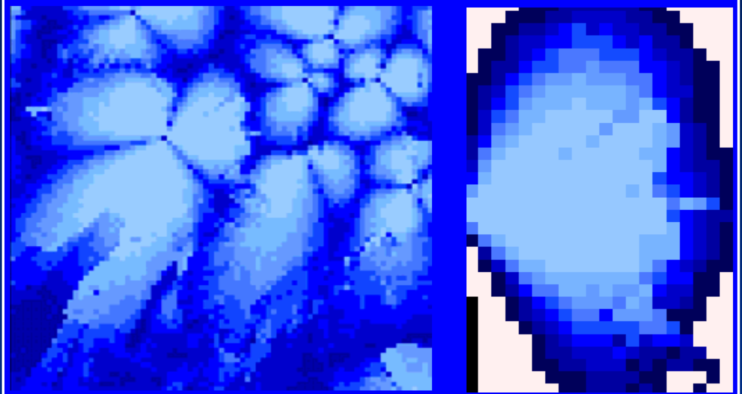
- **sites on regular grid**
- **isotropic propagation conditions**
- **no cell-overlapping**

Best Server Model



- **realistic propagation conditions**
- **arbitrary cell shapes**
- **no cell-overlapping**

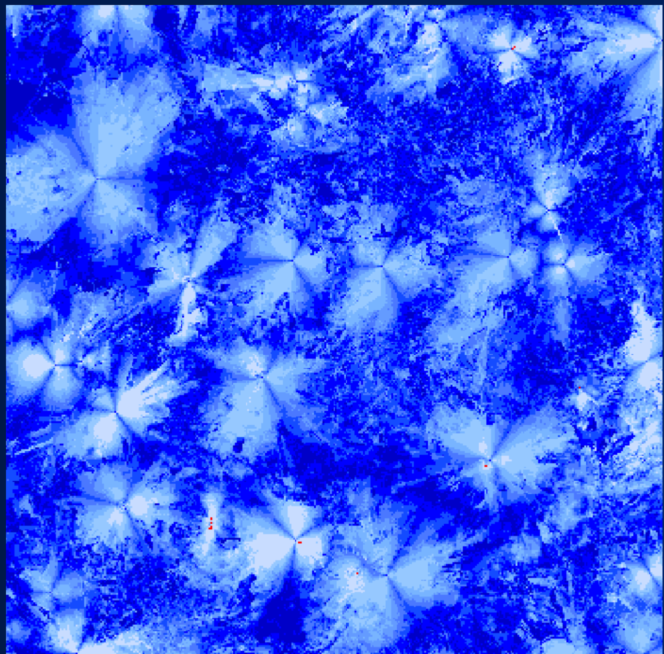
Cell Assignment Probability Model



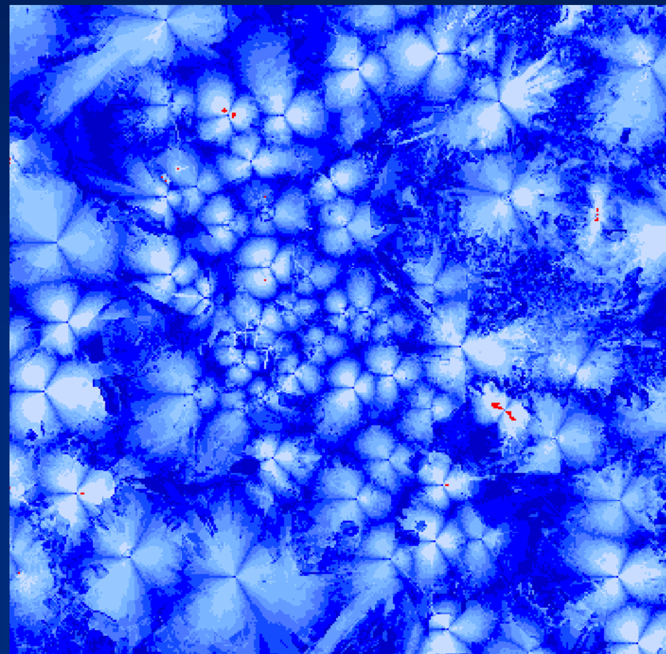
- **realistic propagation conditions**
- **arbitrary cell shapes**
- **cell-overlapping**

Source: **E-Plus Mobilfunk, Germany**

Cell Diagrams



Rural
Terrain Data



Metropolitan
Buildings 3D

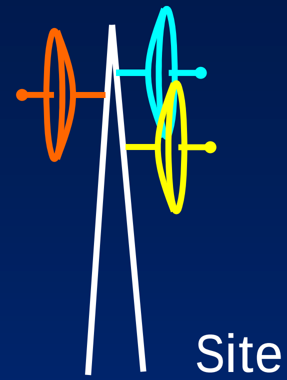
Interference

Level of interference depends on

- distance between transmitters,
- geographical position,
- power of the signals,
- direction in which signals are transmitted,
- weather conditions
- assigned frequencies
  - co-channel interference
  - adjacent-channel interference

Separation

Frequencies assigned to the same location (site) have to be separated



Blocked channels

Restricted spectrum at some locations:

- government regulations,
- agreements with operators in neighboring regions,
- requirements of military forces,
- etc.

Frequency Planning Problem

Find an assignment of frequencies to transmitters that satisfies

– all separation constraints

– all blocked channels requirements

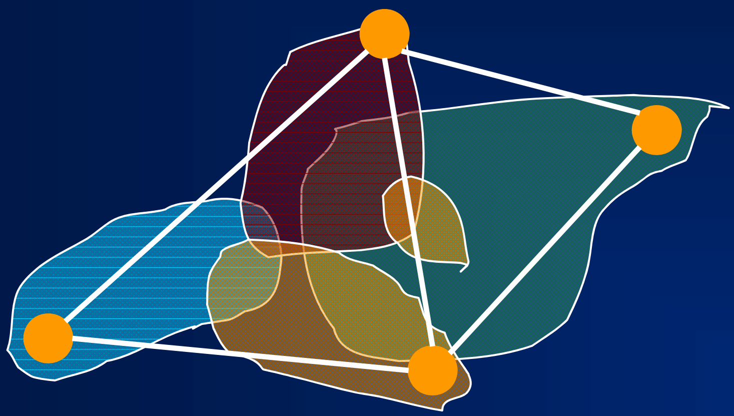
and either

– avoids interference at all

or

– minimizes the (total/maximum) interference level

Modeling: the interference graph



- Vertices represent transmitters (TRXs)
- Edges represent separation constraints and co/adjacent-channel interference
  - Separation distance: $d(vw)$
  - Co-channel interference level: $c^{co}(vw)$
  - Adjacent-channel interference level: $c^{ad}(vw)$

Graph Coloring

Simplifications:

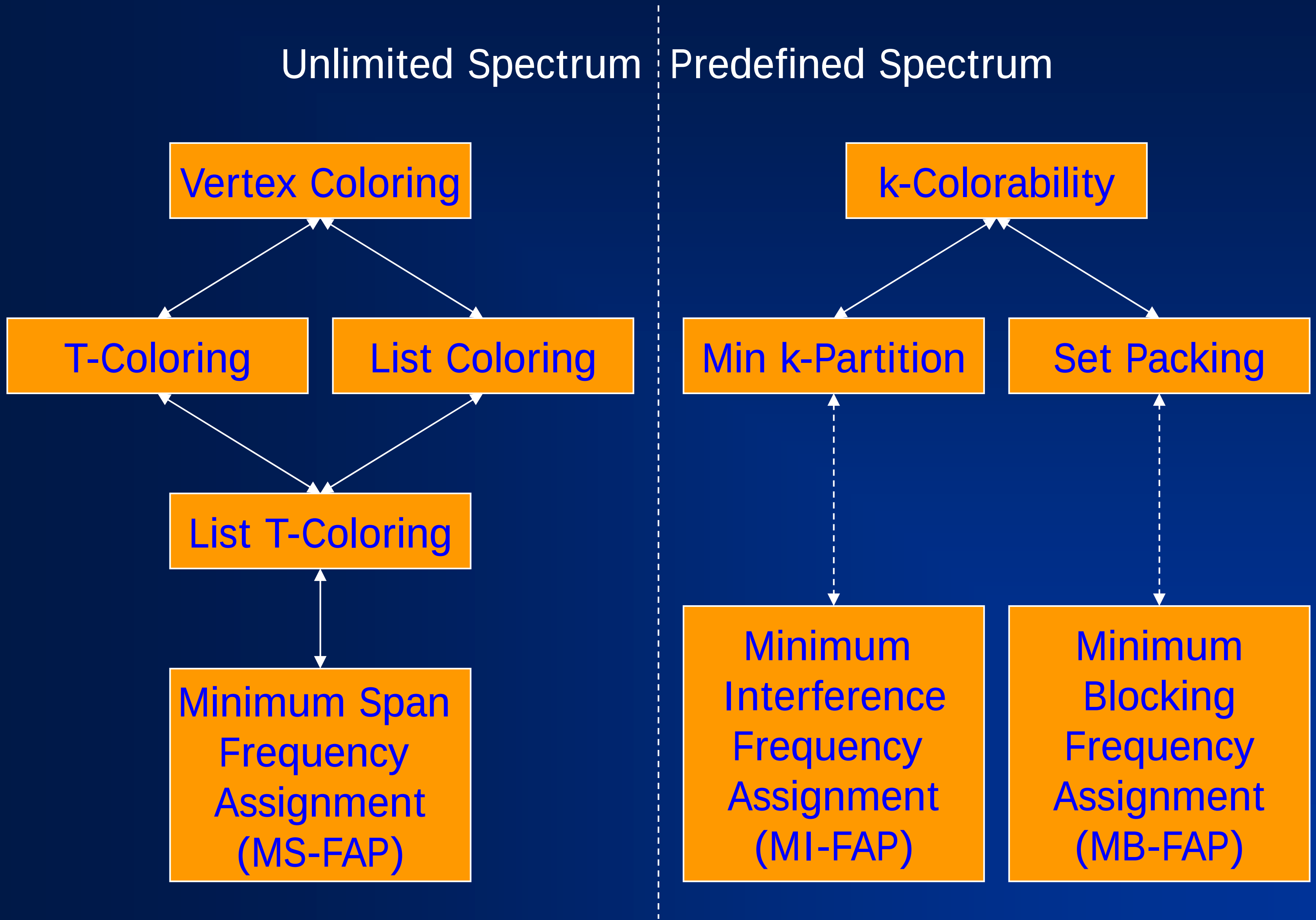
- drop adjacent-channel interference
- drop local blockings
- reduce all separation requirements to 1
- change large co-channel interference into separation distance 1 (inacceptable interference)

Result:

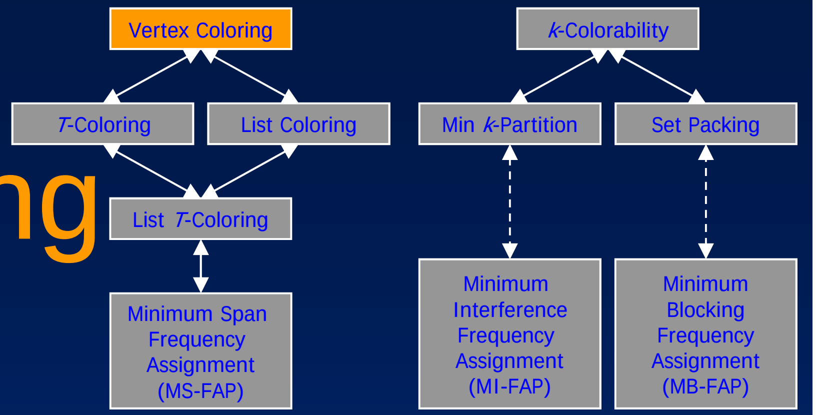
- FAP reduces to coloring the vertices of a graph

Coloring Radio Waves

Graph Coloring & Frequency Planning

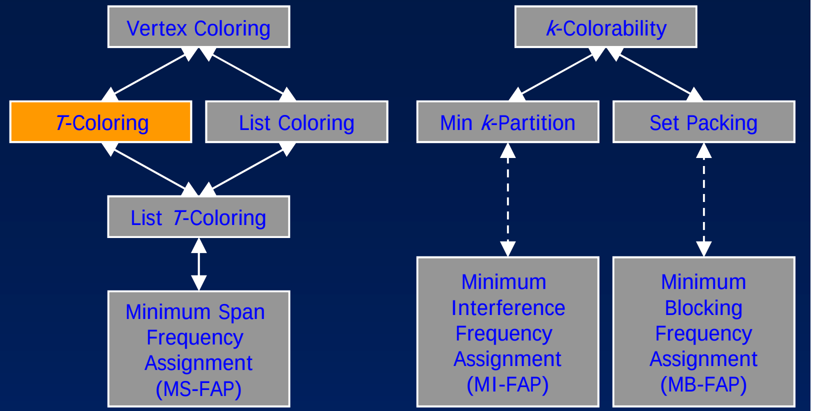


FAP & Vertex Coloring



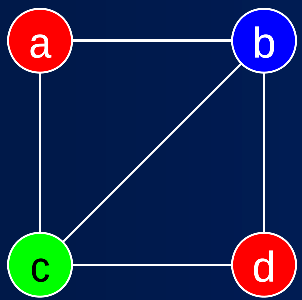
- Only co-channel interference
- Separation distance 1
- Minimization of
  - Number of frequencies used (chromatic number)
  - Span of frequencies used
- Objectives are equivalent: span = #colors-1
- FAP is **NP**-hard

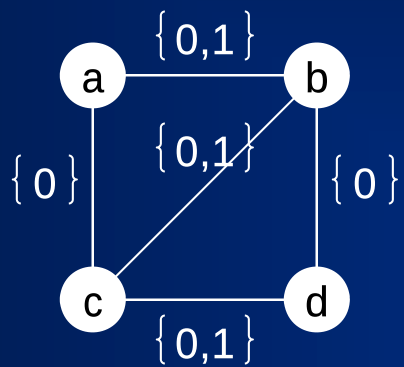
FAP & T-Coloring

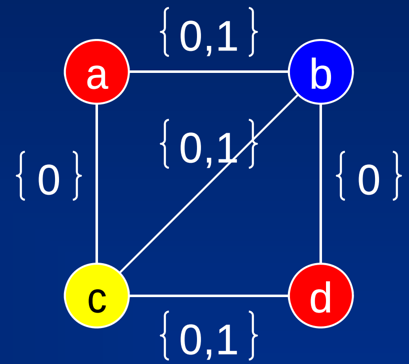


Sets of forbidden distances T_{vw}

$$|f_v - f_w| \notin T_{vw} \qquad T_{vw} = \{0, \ldots, d(vw)\text{-}1\}$$



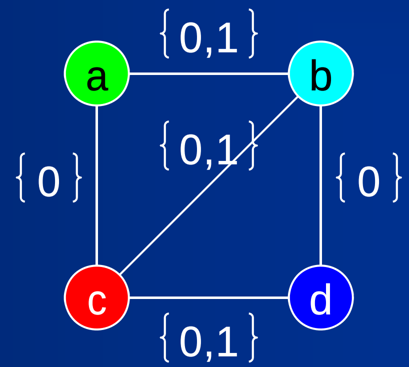




Colors: 3
Span: 4

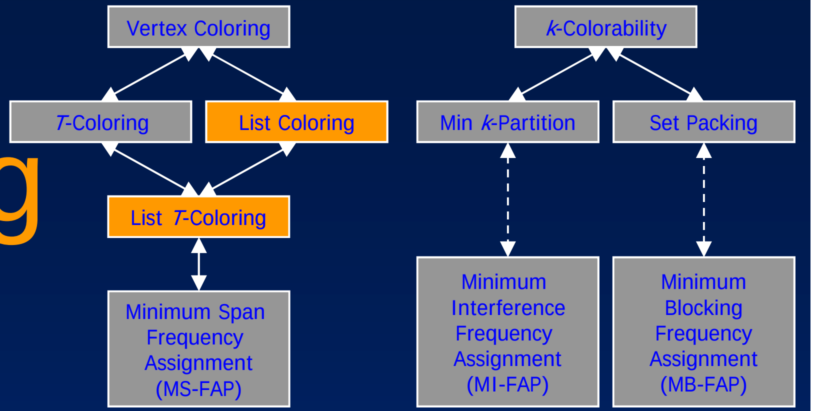
1
2
3
4
5

Minimization of
number of colors and span
are not equivalent!

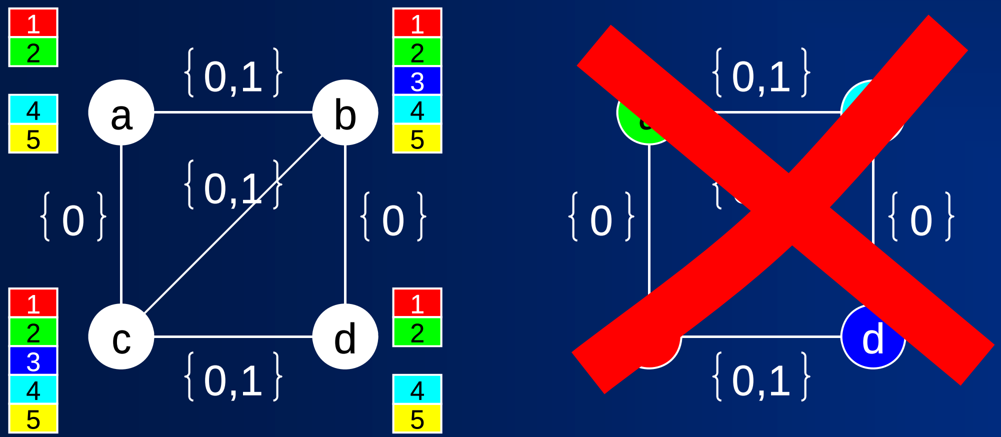


Colors: 4
Span: 3

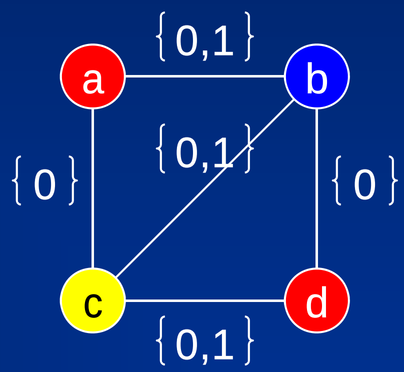
FAP & List- *T*-Coloring



Locally blocked channels:
Sets of forbidden colors B_v

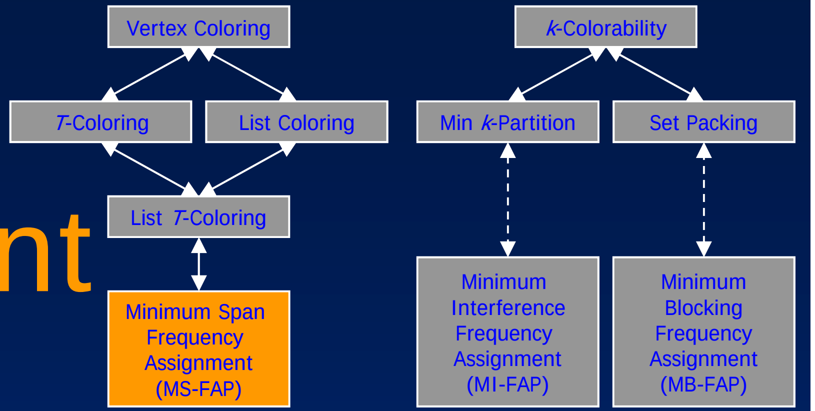


No solution with span 3 !



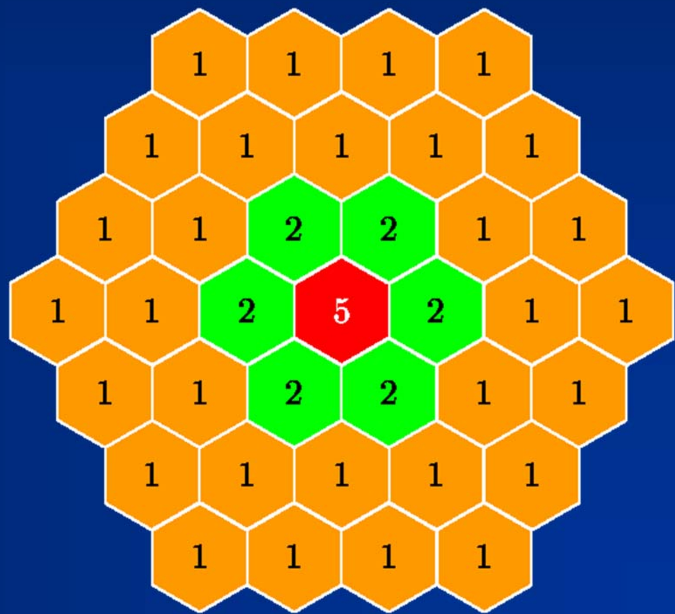
Colors: 3
Span: 4

Minimum Span Frequency Assignment



- List-T-Coloring (+ multiplicity)
- Benchmarks: Philadelphia instances

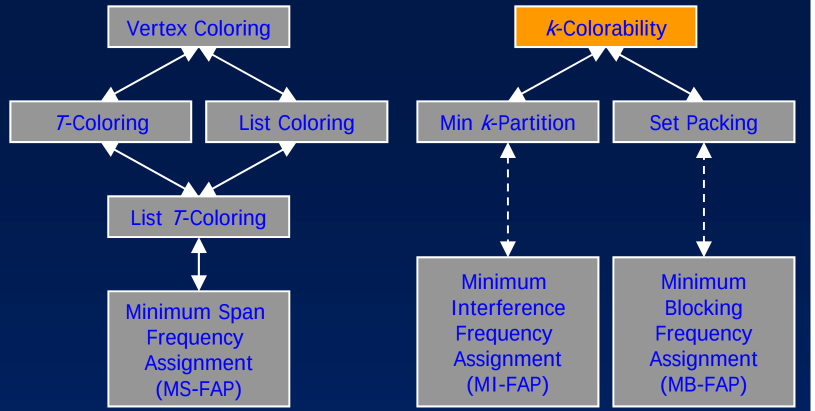
Channel requirements (P1)
Optimal span = 426

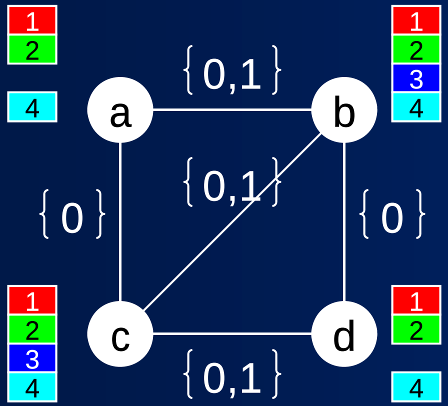


Separation distances

Martin Grötschel

Fixed Spectrum





License for frequencies {1,...,4}

No solution with span 3

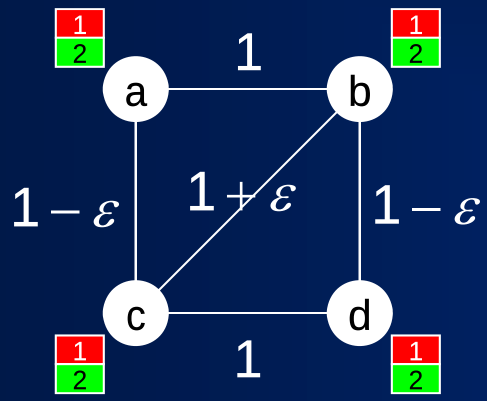
- Is the graph span-*k*-colorable?
- Complete assignment: minimize interference
- Partial assignment without interference

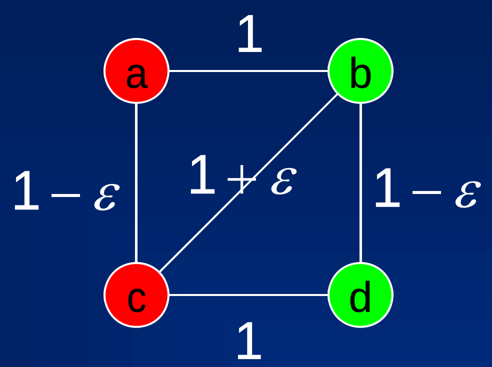
Hard & Soft constraints

- How to evaluate "infeasible" plans?
  - Hard constraints: separation, local blockings
  - Soft constraints: co- and adjacent-channel interference
- Measure of violation of soft constraints: penalty functions

$$p_{vw}(f,g) = \begin{cases} c^{co}(vw) & \text{if } f = g \\ c^{ad}(vw) & \text{if } |f-g| = 1 \\ 0 & \text{otherwise} \end{cases}$$

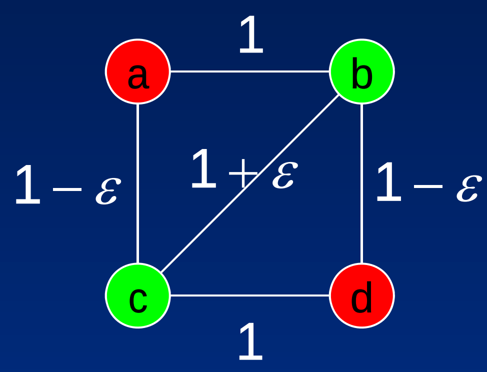
Evaluation of infeasible plans





Total penalty: $2-2\varepsilon$

Maximum penalty: $1-\varepsilon$



Total penalty : $1+\varepsilon$

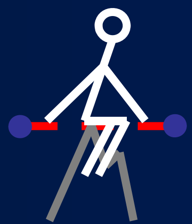
Maximum penalty : $1+\varepsilon$

- Minimizing total interference
- Minimizing maximum interference
  - Use of threshold value, binary search

What is a good objective?

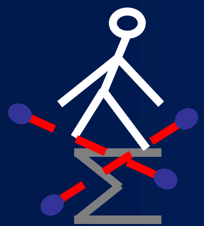
Keep interference information!

Use the available spectrum!



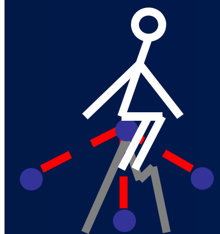
Minimize max interference

T-coloring (min span): Hale; Gamst; ...



Minimize sum over interference

Duque-Anton et al.; Plehn; Smith et al.; ...



Minimize max “antenna” interference

Fischetti et al.; Mannino, Sassano

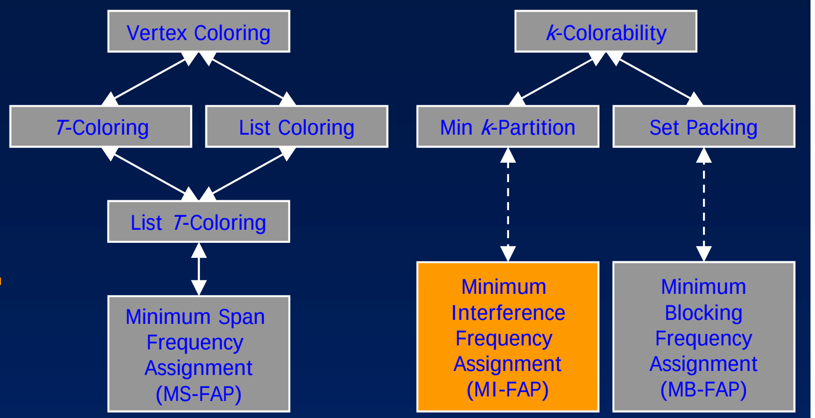
Our Model

Carrier Network:

$$N = (V, E, C, \{B_v\}_{v \in V}, d, c^{co}, c^{ad})$$

- (V,E) is an undirected graph
- C is an interval of integers (spectrum)
- $B_v \subseteq C$ for all $v \in V$ (blocked channels)
- $d : E \rightarrow Z_+$ (separation)
- $c^{co}, c^{ad} : E \rightarrow [0,1]$ (interference)

Minimum Interference Frequency Assignment



Integer Linear Program:

$$\min \quad \sum_{vw \in E^{co}} c_{vw}^{co} z_{vw}^{co} + \sum_{vw \in E^{ad}} c_{vw}^{ad} z_{vw}^{ad}$$

$$s.t. \quad \sum_{f \in F_v} x_{vf} = 1 \qquad \forall v \in V$$

$$x_{vf} + x_{wg} \leq 1 \qquad \forall vw \in E^{d}, |f-g| < d(vw)$$

$$x_{vf} + x_{wf} \leq 1 + z_{vw}^{co} \qquad \forall vw \in E^{co}, f \in F_v \cap F_w$$

$$x_{vf} + x_{wg} \leq 1 + z_{vw}^{ad} \qquad \forall vw \in E^{ad}, |f-g| = 1$$

$$x_{vf}, z_{vw}^{co}, z_{vw}^{ad} \in \{0,1\}$$

Benchmarks

http://fap.zib.de

- Philadelphia (Minimum Span)
- CALMA (MS-FAP, MO-FAP, MI-FAP)

  100-916 transmitters, 40 frequencies, density 5%, “equality” constraints
- COST 259 (MI-FAP)

  2214 transmitters, 75 frequencies, graph density 13.5%, Maximum degree 916, Maximum clique size 93

A Glance at some Instances

Instance	|V|	density [%]	minimum degree	average degree	maximum degree	diameter	clique number
k	267	56,8	2	151,0	238	3	69
B-0-E-20	1876	13,7	40	257,7	779	5	81
f	2786	4,5	3	135,0	453	12	69
h	4240	5,9	11	249,0	561	10	130

Expected graph properties: planarity,...

Computational Complexity

Neither high quality nor feasibility are generally achievable within practical running times:

- Testing for feasibility is NP-complete.
- There exists an $\varepsilon > 0$ such that FAP cannot be "approximated" within a factor of $|V|^{\varepsilon}$ unless $P = NP$.

Heuristic Solution Methods

- Greedy coloring algorithms,
- DSATUR,
- Improvement heuristics,
- Threshold Accepting,
- Simulated Annealing,
- Tabu Search,
- Variable Depth Search,
- Genetic Algorithms,
- Neural networks,
- etc.

Heuristics

- T-coloring o
- Dual Greedy --
- DSATUR with Costs ++

construction heuristics

- Iterated 1-Opt o
- Simulated Annealing +
- Tabu-Search o
- Variable Depth Search ++

(randomized) local search

- MCF -
- B&C-based +

other improvement heuristics

Region with "Optimized Plan"

Instance k, a "toy case" from practice

264 cells

267 TRXs

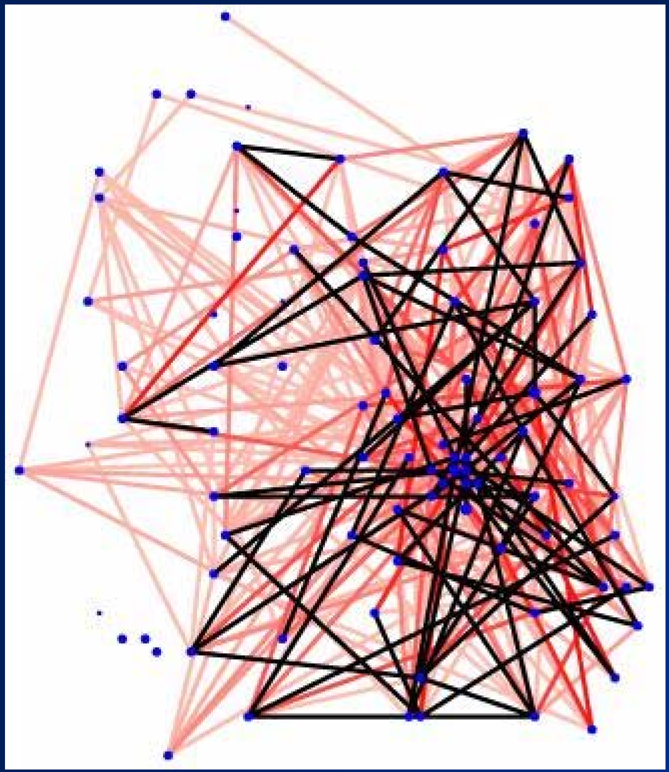
50 channels

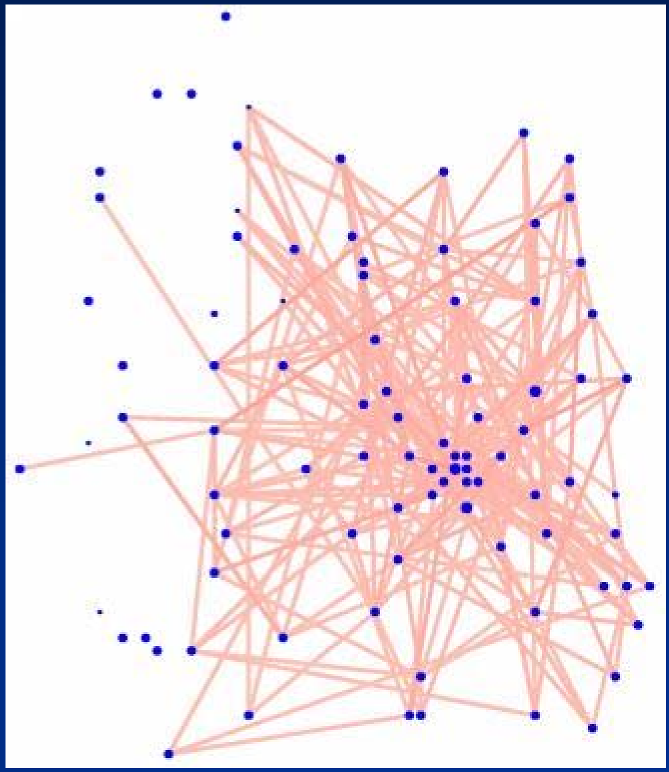
57% density

151 avg.deg.

238 max.deg.

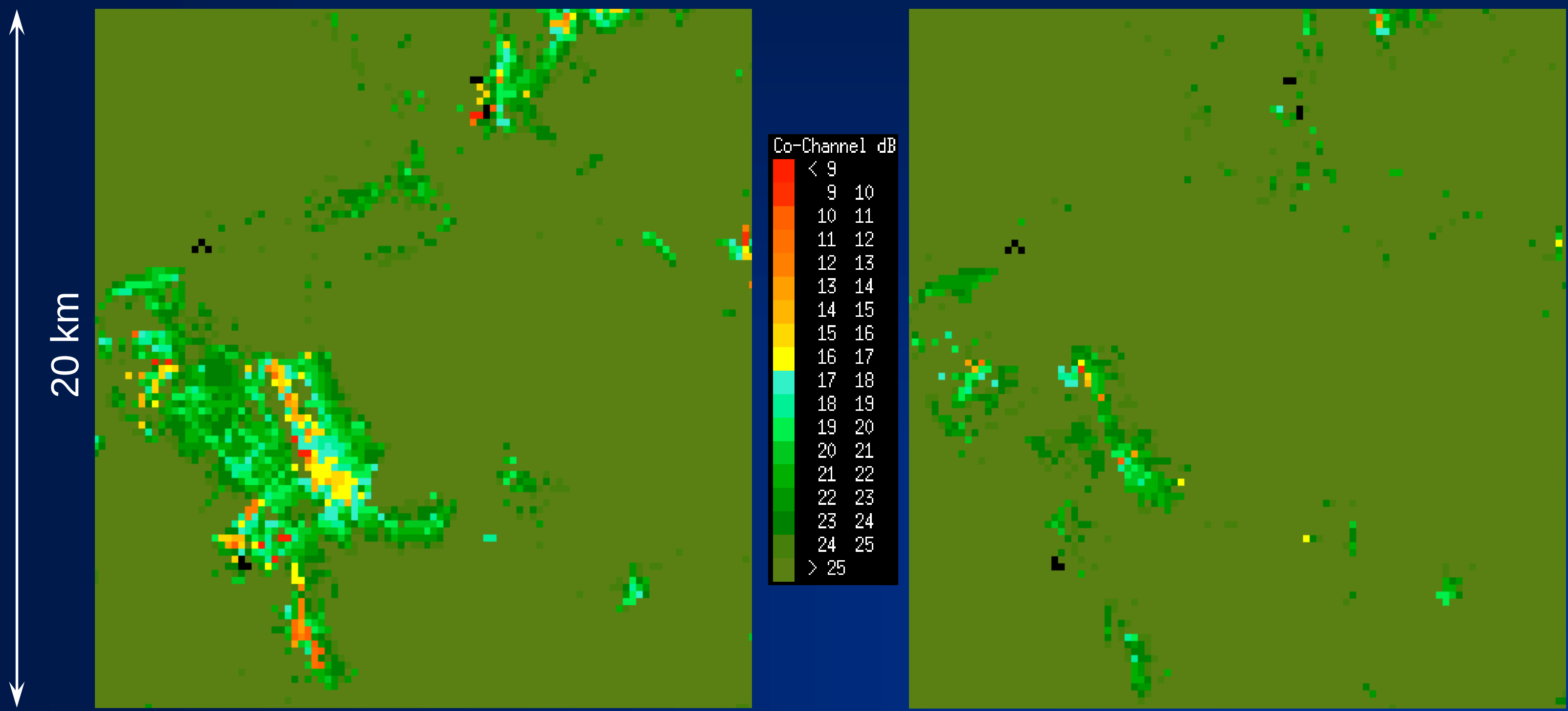
69 clique size





DC5-VDS: Reduction 96,3%

co-channel C/I worst Interferer



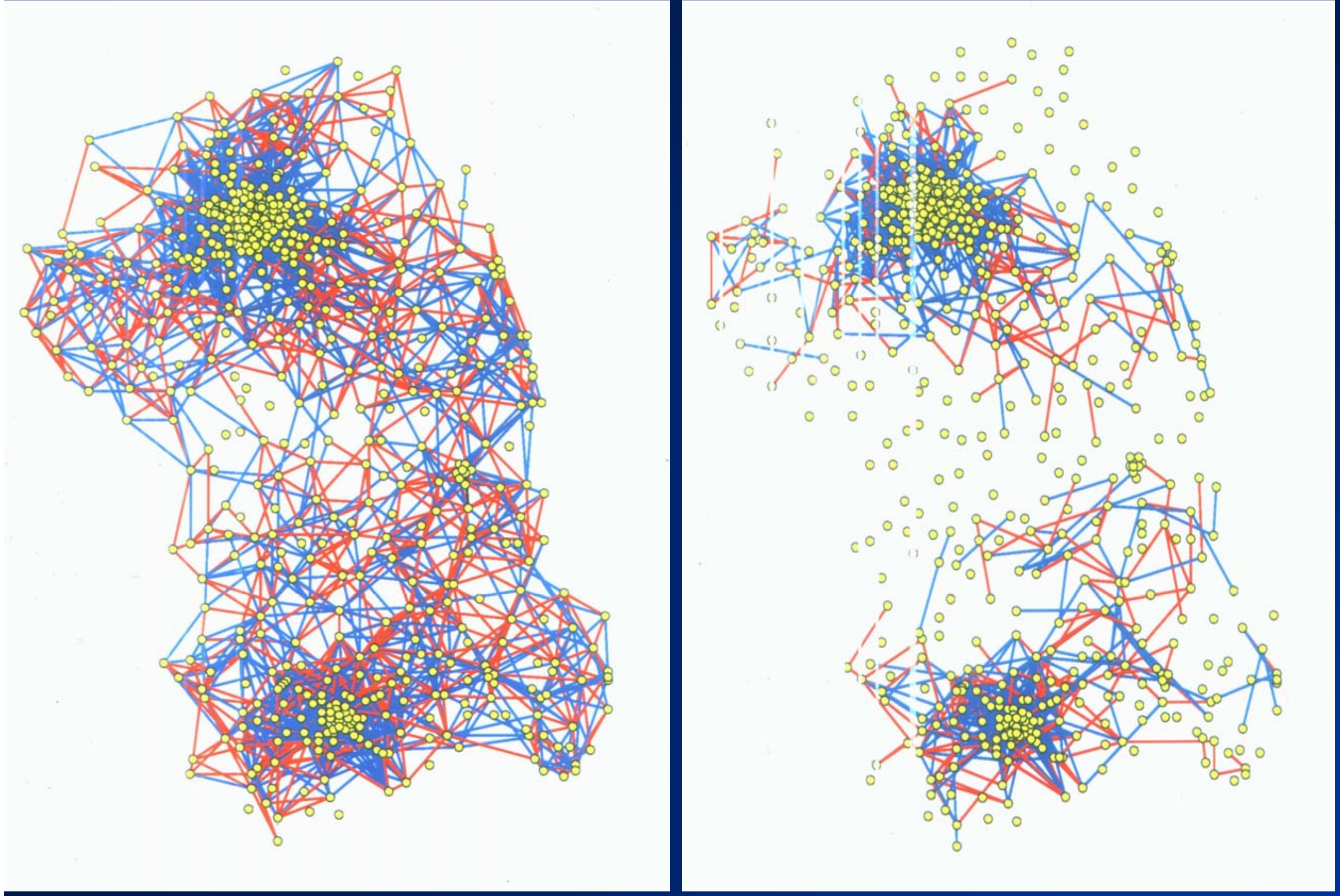
Mobile Systems International Plc.

Mobile Systems International Plc.

Commercial software

DC5-IM

Region Berlin - Dresden

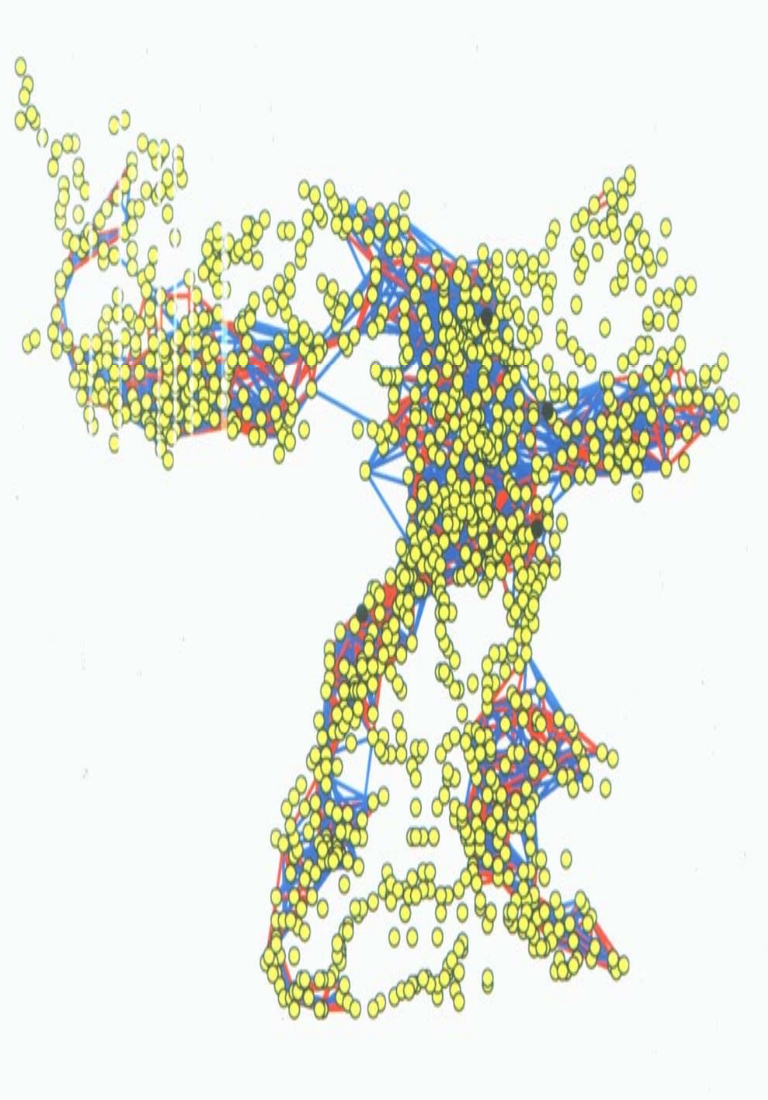


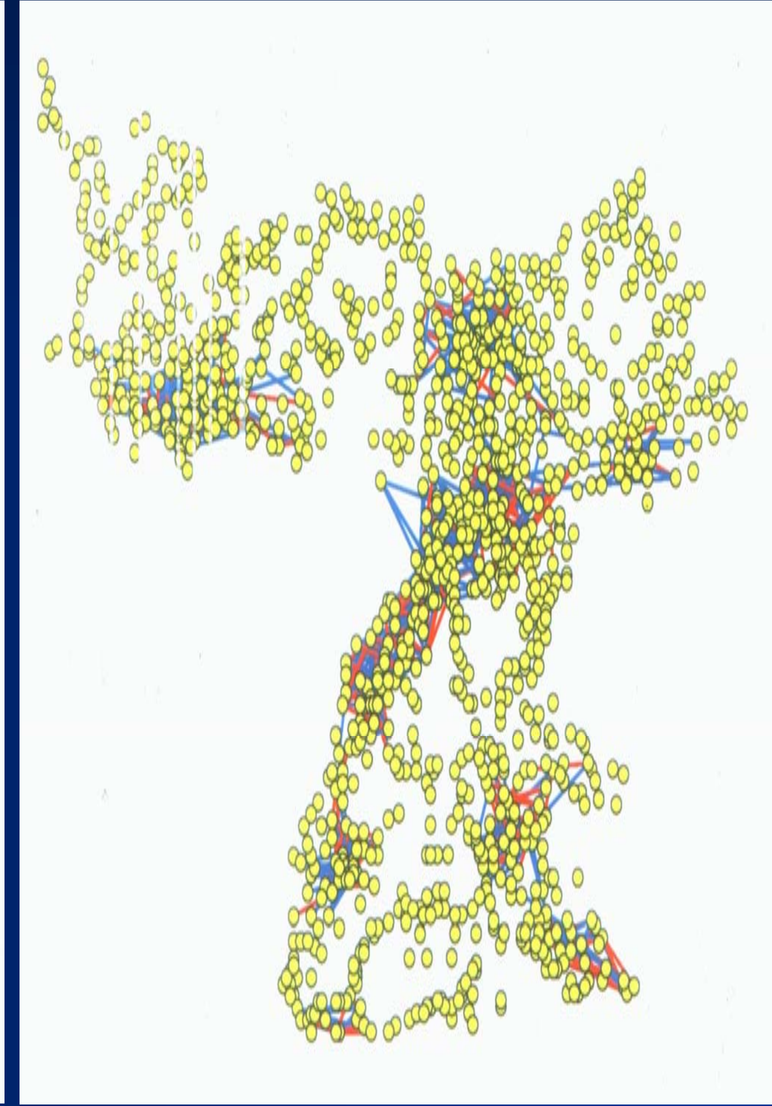
2877
carriers

50 channels

Interference
reduction:
83.6%

Region Karlsruhe





2877
Carriers

75 channels

Interference
Reduction:
83.9 %

Heuristics: Summary

- FAP is NP-hard and hardly approximable - in theory.
- Efficient heuristic optimization allows significant improvement over standard planning methods.
- Generating the right input data is nontrivial.
- SA: solution quality depends heavily on the neighborhood relation and also on the cooling schedule.

Guaranteed Quality

Optimal solutions are out of reach!

Enumeration: $50^{267} \approx 10^{197}$ combinations (for trivial instance k)

Hardness of approximation

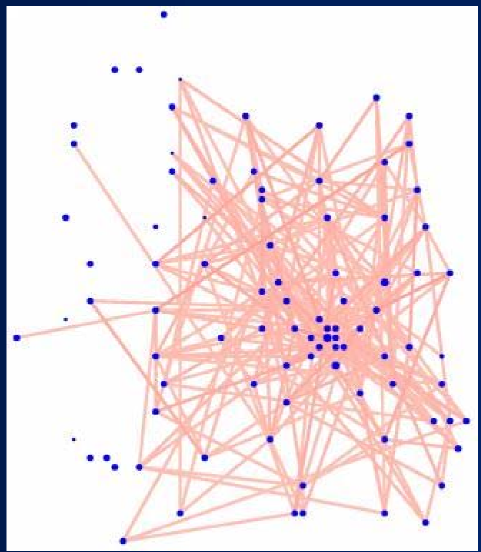
Polyhedral investigation (IP formulation)

Aardal et al.; Koster et al.; Jaumard et al.; …

Used for adapting to local changes in the network

Lower bounds - study of relaxed problems

Lower Bounds



Instance k

How much better can we possibly be?

Lower Bounding Technology

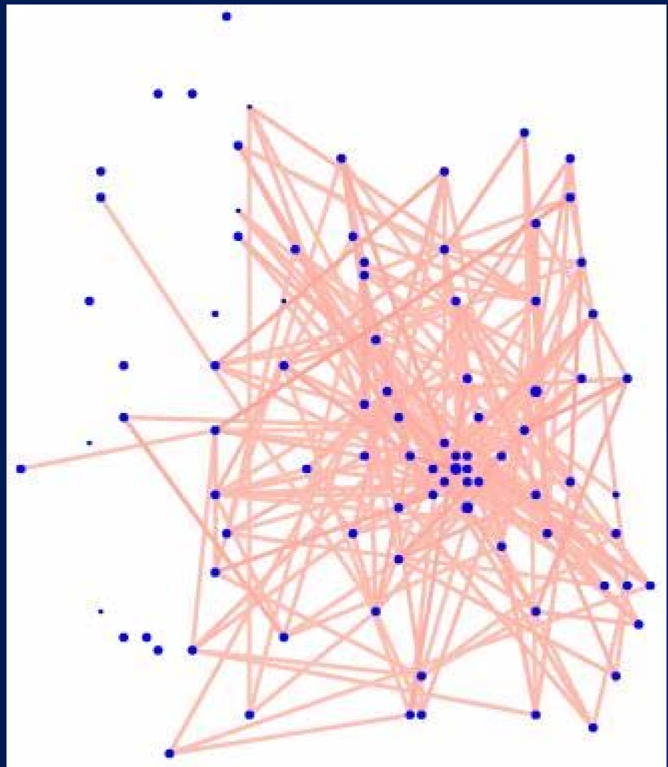
- LP lower bound for coloring
- TSP lower bound for *T*-coloring
- LP lower bound for minimizing interference
- Tree Decomposition approach
- Semidefinite lower bound for minimizing interference

Region with “Optimized Plan”

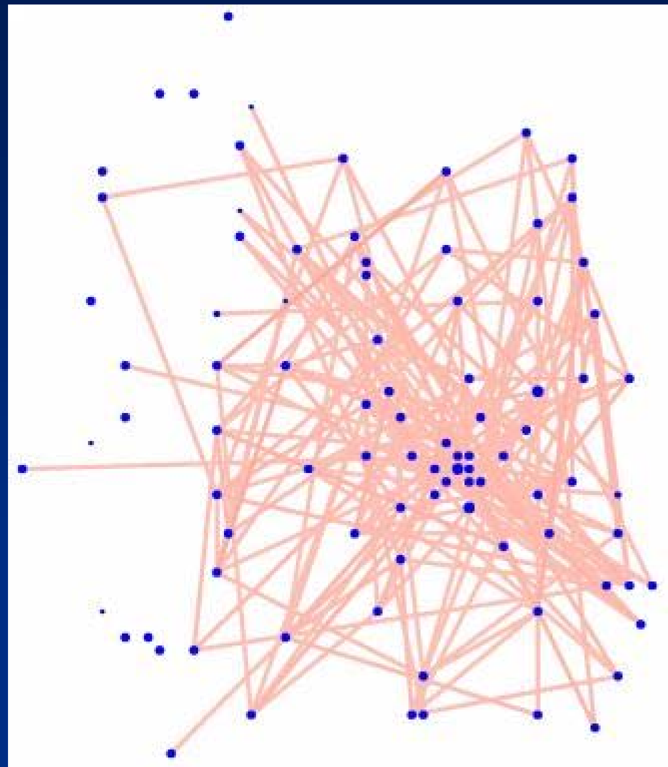
Instance k, the “toy case” from practice

264 cells
267 TRXs
50 channels

57% density
151 avg.deg.
238 max.deg.
69 clique size

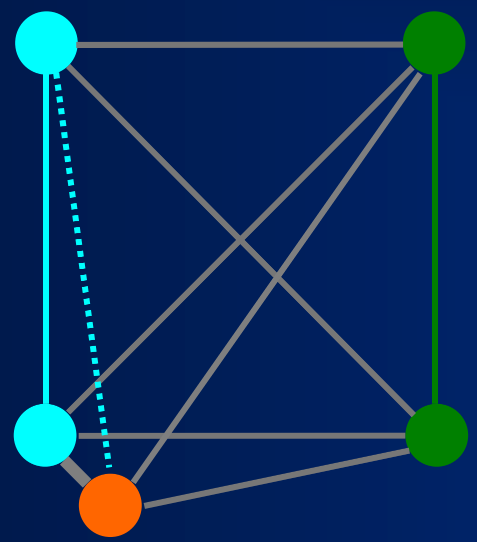


DC5-VDS



Further Reduction: 46.3%

Lower Bounds



5-clique & 3 channels

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two interfering pairs

- *Weak* Clique Bound:
  add cheapest edges of clique
- Clique Bound:
  optimal assignment on clique
- Clique-Cover Bound:
  max sum clique bounds
- Min k-Partition:
  optimal assignment for simplified network

A Simplification of our Model

Simplified Carrier Network:

$$N = (V, E, C, \{B_v\}_{v \in V}, d, c^{co}, c^{ad})$$

- (V,E) is an undirected graph
- C is an interval of integers (spectrum)
- $B_v \subseteq C$ for all $v \in V$ (blocked channels)
- $d : E \to Z_+$ {0, 1} (separation)
- $c^{co}, c^{ad} : E \to [0,1]$ (interference)

MIN K-Partition

- No blocked channels
- No separation constraints larger than one
- No adjacent-channel interference

min k-partition (max k-cut)

Chopra & Rao; Deza et al.; Karger et al.; Frieze & Jerrum

IP, LP-based B&C, SDP

MIN K-Partition

Given: an undirected graph $G = (V,E)$ together with real edge weights w_{ij} and an integer k.

Find a partition of the vertex set into (at most) k sets $V_1, \ldots, V_k$ such that the sum of the edge weights in the induced subgraphs is minimal!

$$\min_{\substack{V_1,\ldots,V_k \\ \text{partition of } V}} \sum_{p=1}^{k} \sum_{i,j \in V_p} w_{ij}$$

NP-hard to approximate optimal solution value.

Integer Linear Programmming

(ILP)

$$\min \sum_{i,j \in V} w_{ij} z_{ij}$$

$$z_{ih} + z_{hj} - z_{ij} \leq 1 \qquad \forall h, i, j \in V$$ -> partition consistent

$$\sum_{i,j \in Q} z_{ij} \geq 1 \qquad \forall\, Q \subseteq V \text{ with } |Q| = k + 1$$ -> use at most k blocks

$$z_{ij} \in \{0, 1\}$$

Number of ILP inequalities (facets)

Instance*	|V|	k	Triangle	Clique Inequalities
cell.k	**69**	**50**	**157182**	**17231414395464984**
B-0-E	81	75	255960	25621596
B-1-E	**84**	**75**	**285852**	**43595145594**
B-2-E	93	75	389298	1724861095493098563
B-4-E	120	75	842520	1334655509331585084721199905599180
B-10-E	174	75	2588772	361499854695979558347628887341189586948364637617230

Vector Labeling

Lemma: For each k, n ($2 \leq k \leq n+1$) there exist k unit vectors $u_1, \ldots, u_k$ in n-space, such that their mutual scalar product is $-1/(k-1)$. (This value is least possible.)

Fix $U = \{u_1, \ldots, u_k\}$ with the above property, then the min k-partition problem is equivalent to:

$$\min_{\substack{\phi:V\to U \\ i\mapsto\phi_i}} \sum_{ij\in E} \left(\frac{k-1}{k}\langle\phi_i,\phi_j\rangle + \frac{1}{k}\right) w_{ij}$$

$X = [\langle\phi_i, \phi_j\rangle]$ is positive semidefinite, has 1's on the diagonal, and the rest is either $-1/(k-1)$ or 1.

Semidefinite Relaxation

(SDP)

$$\min \sum_{ij \in E(K_n)} w_{ij} \frac{(k-1)\,V_{ij}+1}{k}$$

$$V_{ii} = 1 \qquad \forall i \in V$$

$$V_{ij} \geq \frac{-1}{k-1} \qquad \forall i,j \in V$$

$$V \succeq 0$$

Solvable in polynomial time!

Given V, let $z_{ij} := ((k-1)\ V_{ij} + 1)/k$, then:

- z_{ij} in [0,1]
- $z_{ih} + z_{ih} - z_{ij} < \sqrt{2}$ (<=1)
- $\sum_{i,j \text{ in } Q} z_{ij} > ½$ (>=1)

(SDP) is an approximation of (ILP)

Computational Results

S. Burer, R.D.C Monteiro, Y. Zhang; Ch. Helmberg; J. Sturm

Instance	clique cover	min k-part.	*heuristic*	clique cover	min k-part.	*heuristic*
cell.k	**0,0206**	**0,0206**	***0,0211***	**0,0248**	**0,1735**	**0,4023**
B-0-E	0,0016	0,0013	*0,0016*	0,0018	0,0096	*0,8000*
B-1-E	**0,0063**	**0,0053**	***0,0064***	**0,0063**	**0,0297**	**0,8600**
B-2-E	0,0290	0,0213	*0,0242*	0,0378	0,4638	*3,1700*
B-4-E	0,0932	0,2893	*0,3481*	0,2640	4,3415	*17,7300*
B-10-E	0,2195	2,7503	*3,2985*			*146,2000*

maximal clique (first three value columns) — entire scenario (last three value columns)

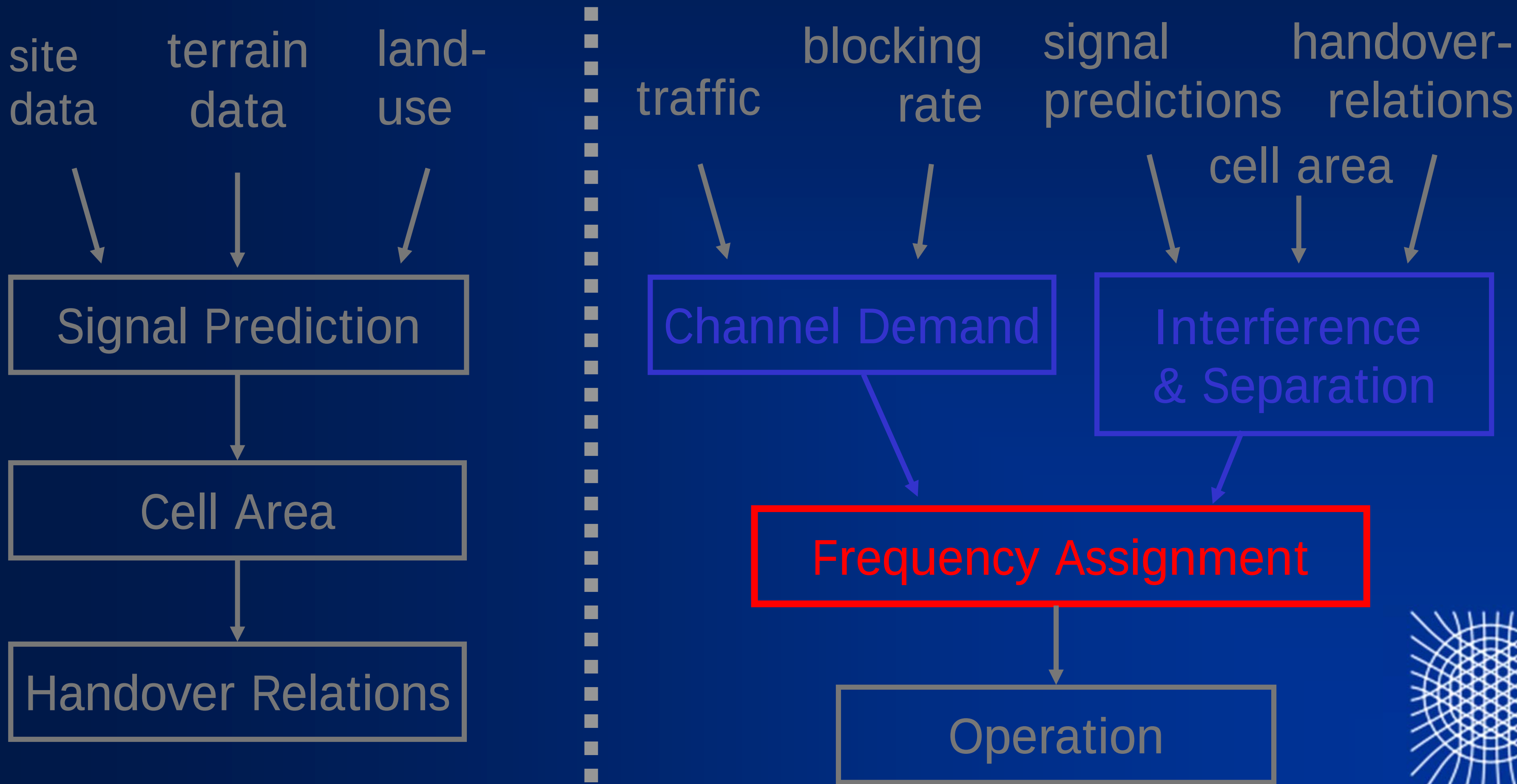
Lower bound on co-channel interference by a factor of 2 to 85 below co- and adjacent-channel interference of best known assignment.

Semidefinite Conclusions

Lower bounding via
Semidefinite Programming works,
at least better than LP!

- Challenging computational problems
- Bounds too far from cost of solutions to give strong quality guarantees
- How to produce good k-partitions starting from SDP solutions?

Summary: Radio Planning



Mathematical Approach: Summary

- Collecting sound data is intricate.
- Minimizing the sum of interference is a compromise with practical under-pinning.
- Huge improvements are possible.
- Practice: time-savings, increased quality!

Optimization really helps!

Literature (ZIB PaperWeb)

K. Aardal, S. van Hoesel, A. Koster, C. Mannino, A. Sassano, “Models and Solution Techniques for the Frequency Assignment Problem”, ZIB-report 01-40, 2001.

A. Eisenblätter, “Frequency Assignment in GSM Networks: Models, Heuristics, and Lower Bounds”, Ph.D. thesis TU Berlin, 2001.

A. Eisenblätter, M. Grötschel, A. Koster, “Frequency Planning and Ramifications of Coloring”, Discussiones Mathematicae, Graph Theory, 22 (2002) 51-88.

A. Eisenblätter, M. Grötschel, A. Koster, "Frequenzplanung im Mobilfunk", DMV-Mitteilungen 1/2002, 18-25

A. Koster, “Frequency Assignment – Models and Algorithms”, Ph.D. thesis Universiteit Maastricht, 1999.

“FAP web – A website devoted to Frequency Assignment”, http://fap.zib.de, 2000.

The End

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